

## On the scaling and ageing behaviour of the alternating susceptibility in spin glasses and local scale invariance

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2005 J. Phys.: Condens. Matter 17 S1899

(<http://iopscience.iop.org/0953-8984/17/20/017>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 129.252.86.83

The article was downloaded on 28/05/2010 at 04:52

Please note that [terms and conditions apply](#).

# On the scaling and ageing behaviour of the alternating susceptibility in spin glasses and local scale invariance

Malte Henkel<sup>1</sup> and Michel Pleimling<sup>2</sup>

<sup>1</sup> Laboratoire de Physique des Matériaux<sup>3</sup>, Université Henri Poincaré Nancy I, BP 239, F-54506 Vandœuvre lès Nancy Cedex, France

<sup>2</sup> Institut für Theoretische Physik I, Universität Erlangen-Nürnberg, Staudtstraße 7B3, D-91058 Erlangen, Germany

Received 18 December 2004, in final form 10 January 2005

Published 6 May 2005

Online at [stacks.iop.org/JPhysCM/17/S1899](http://stacks.iop.org/JPhysCM/17/S1899)

## Abstract

The frequency-dependent scaling of the dispersive and dissipative parts of the alternating susceptibility is studied for spin glasses at criticality. An extension of the usual  $\omega t$ -scaling is proposed. Simulational data from the three-dimensional Ising spin glass agree with this new scaling form and moreover reproduce well the scaling functions explicitly calculated for systems satisfying local scale invariance. There is also a qualitative agreement with existing experimental data.

(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

Understanding the complex behaviour of glass-forming systems cooled to below their glass transition temperature remains a challenge. At first sight, the problem might appear to be hopelessly difficult, since time-translation invariance is in general broken and the properties of observables may hence depend on the prehistory of the material under study (i.e. thermal, mechanical, . . .). On the other hand, an important discovery has been the observation of *dynamical scaling*, see [1], which occurs quite independently of whether the equilibrium state is critical or not. In recent years, it has been realized that many aspects of dynamical scaling are conveniently first studied in non-disordered, i.e. ferromagnetic systems. After a quench to or to below their critical temperature  $T_c$ , these systems undergo an ageing behaviour which in many respects is quite similar to the one in glassy or kinetically constrained systems. The manifold problems which arise in the study of ageing in simple magnets or glasses are reviewed, e.g., in [2–9].

<sup>3</sup> Laboratoire associé au CNRS UMR 7556.

For notational simplicity, we shall in what follows consider magnetic spin systems. Convenient tools for the study of ageing behaviour are the two-time autocorrelation and autoresponse functions

$$C(t, s) = \langle \phi(t)\phi(s) \rangle \sim s^{-b} f_C(t/s) \quad (1)$$

$$R(t, s) = \left. \frac{\delta \langle \phi(t) \rangle}{\delta h(s)} \right|_{h=0} \sim s^{-1-a} f_R(t/s) \quad (2)$$

where  $\phi(t)$  is the order parameter at time  $t$  and  $h(s)$  is the conjugate magnetic field at time  $s$ . The scaling behaviour is expected to apply in the so-called *ageing regime* where  $t, s \gg t_{\text{micro}}$  and  $t - s \gg t_{\text{micro}}$ , where  $t_{\text{micro}}$  is a microscopic timescale. Furthermore, we tacitly assumed that the scaling derives from the time dependence of a single characteristic length scale  $L(t)$  which measures the linear size of correlated clusters. In this paper, we shall always consider algebraic growth laws, namely  $L(t) \sim t^{1/z}$ , where  $z$  is the dynamic exponent. Then the above forms define the nonequilibrium exponents  $a$  and  $b$  and the scaling functions  $f_C(y)$  and  $f_R(y)$ . For large arguments  $y \rightarrow \infty$ , one generically expects

$$f_C(y) \sim y^{-\lambda_C/z}, \quad f_R(y) \sim y^{-\lambda_R/z} \quad (3)$$

where  $\lambda_C$  and  $\lambda_R$ , respectively, are known as autocorrelation [10, 11] and autoresponse exponents [12]. This description applies to many simple magnets quenched to a temperature  $T \leq T_c$  as is well known; see [2, 3, 5]<sup>4</sup>. On the other hand, for glasses quenched to below the glass transition temperature, a slow crossover between an algebraic growth law at short times and a slower (logarithmic) growth at larger times appears to give a better description of the data; see [8] and references therein. Recently, evidence was found that for spin glasses quenched to their critical temperature  $T = T_c$  a simple scaling of the two-time observables compatible with an algebraic growth law applies [13].

In recent years, from the study of ageing in simple magnets it has been proposed that global dynamical scaling might be extended to a local scale invariance [14, 15]. One of the simplest predictions of that theory is the explicit form of the two-time autoresponse function. It reads<sup>5</sup>

$$R(t, s) = s^{-1-a} f_R(t, s), \quad f_R(y) = f_0 y^{1+a'-\lambda_R/z} (y-1)^{-1-a'} \quad (4)$$

where  $a'$  is a new independent exponent and  $f_0$  is a normalization constant. The independent existence of the exponent  $a'$  was recognized recently [17] for  $z = 2$  and we shall extend that argument to arbitrary  $z$  in appendix A. Previous derivations of  $f_R(y)$  [14, 15] had assumed  $a = a'$  from the outset. The only known example with distinct exponents  $a \neq a'$  was the 1D Glauber–Ising model at temperature  $T = 0$  and with initial power-law correlations of the form  $\langle \sigma_i \sigma_j \rangle \sim |i - j|^{-\nu}$  with  $\nu \geq 0$ . The exact solution for  $R(t, s)$  [18–20] is of the form (4) and one can read off  $a = 0$ ,  $a' = -\frac{1}{2}$  and  $\lambda_R = 1$  [17]. In appendix B, we give a second example and show that the treatment of phase ordering in  $d$  spatial dimensions at zero temperature in the spirit of the OJK approximation [21, 22] leads to

$$a_{\text{OJK}} = \frac{d-1}{2}, \quad a'_{\text{OJK}} = \frac{d-2}{2}, \quad \lambda_{R,\text{OJK}} = \frac{d}{2}. \quad (5)$$

The same scaling function and hence (5) is also found from the Gaussian approximation to phase ordering kinetics [23]<sup>6</sup>.

<sup>4</sup> Exceptions occur for example in the 2D  $XY$  model with a fully disordered initial state.

<sup>5</sup> Technically, this requires that the order parameter is a *quasi-primary* field under local scale transformations [14]. This concept is the analogue of one used in conformal field theory [16].

<sup>6</sup> We observe that in both examples  $a = a' + \frac{1}{2}$  but it is still open to what extent this might be a general relationship.

Equation (4) with  $a = a'$  is perfectly reproduced in simple magnets undergoing coarsening after a quench to  $T < T_c$ , most notably the 2D/3D Glauber–Ising model [15, 24], the 3D  $XY$  model [25] and in several exactly solvable systems [14, 27, 17]<sup>7</sup>. For critical quenches with  $T = T_c$ , the agreement between numerical data and (4) with  $a = a'$  is almost perfect in the 2D/3D Ising and  $XY$  models [15, 27, 25, 28] and the 1D contact process [29] and is exact in several exactly solvable models with  $z = 2$  [14, 12, 17]. On the other hand, a second-order  $\varepsilon$ -expansion calculation from renormalized field theory gives a small but systematic deviation with respect to equation (4) [9, 30].

Of course it would be interesting to see whether a scaling description or even an extension to local scale invariance might be applicable to glassy systems as well. Such a comparison may be far from straightforward, however. Indeed, a possible scaling behaviour of two-time correlators  $C(t, s)$  and integrated responses  $\int du R(t, u)$  has been discussed for a long time, both theoretically and experimentally, see [1, 3, 6, 4, 31, 8], and debates have arisen over several central issues of which we mention a few. First, for glasses quenched to below their glass temperature, it is not entirely clear whether the growth law for  $L(t)$  is algebraic or logarithmic [32]. Second, even if an asymptotic power-law scaling is accepted, there has been an intense debate on whether scaling occurs according to the so-called ‘full ageing’ scenario, that is in terms of the scaling variable  $y = t/s$  (possibly with small logarithmic corrections) [33], or whether a ‘subageing’ scenario applies, with a scaling variable  $\xi := [t^{1-\mu} - s^{1-\mu}]/(1-\mu)$ , where  $\mu$  is a free parameter [31]. The usual power-law scaling is recovered in the  $\mu \rightarrow 1$  limit, but in many experiments the data are fitted with values of  $\mu$  as low as  $\approx 0.8$ – $0.9$ . It has been suggested recently [34] that values of  $\mu < 1$  merely result from a quench to below the glass transition which is not yet sufficiently rapid, but the repetition of that experiment on other substances has not yet led to unambiguous conclusions [31]. Third, it is no longer even obvious that the commonly studied spin glass models really mimic the experimentally studied materials sufficiently well (in spite of well-established qualitative similarities [35]): recent simulations on 3D/4D Ising and Heisenberg spin glasses provide evidence for cumulative ageing and rejuvenation phenomena in temperature cycling which are not observed in real spin glass materials [36].

In view of these many difficulties, it might be simpler to consider the behaviour of glassy systems from a different point of view. One rather works with a time-dependent (oscillating) magnetic field and studies simultaneously the dependence on time and on the imposed oscillation angular frequency  $\omega$ . For a harmonic magnetic field, it is common to consider the real and the imaginary part of the magnetic susceptibility

$$\begin{aligned}\chi'(\omega, t) &= \int_0^t du R(t, u) \cos(\omega(t-u)) \\ \chi''(\omega, t) &= \int_0^t du R(t, u) \sin(\omega(t-u))\end{aligned}\tag{6}$$

where  $R(t, s)$  is the linear response discussed above. In this setting,  $1/\omega$  provides the second timescale, the natural scaling variable is  $y = \omega t$  and the scaling regime should be reached in the limit  $\omega \rightarrow 0$  and  $t \rightarrow \infty$ . In many experiments and simulations, one averages over at least one period of the oscillating field; see e.g. [35]. Then in a great variety of glass-forming substances quenched to below or to near to their glass transition point one observes good but not always perfect evidence for an  $\omega t$ -scaling behaviour of the following form for the period-averaged

<sup>7</sup> A further extension of local scale invariance with  $z = 2$  in  $d$  spatial dimensions to a new type of conformal invariance in  $d + 2$  dimensions yields a prediction of  $C(t, s)$  which is in agreement with numerical data in the 2D Glauber–Ising model [26].

dissipative (imaginary) part [32, 37]:

$$\overline{\chi''}(\omega, t) = \chi''_{\text{st}}(\omega) + \chi''_{\text{age}}(\omega, t), \quad \chi''_{\text{age}}(\omega, t) \simeq A''_{\text{age}}(\omega t)^{-b''} \quad (7)$$

where  $\chi''_{\text{st}}$  is thought of as a ‘stationary’ contribution while the ageing behaviour is described by  $\chi''_{\text{age}}$ . The amplitude  $A''_{\text{age}}$  and the exponent  $b''$  are obtained from fits to the experimental data but there does not seem to exist a relationship with the exponents  $a, a', b, \lambda_{C,R}$  in the literature.

Similar scaling forms have been proposed for the dispersive (real) part  $\overline{\chi'}$ , but in practice the imaginary part is usually easier to measure. It is usually thought that  $b' = b''$ .

In this paper, we shall try and see whether a relationship between the exponents  $b''$  and  $b'$  of the alternating susceptibilities (6) and the other nonequilibrium exponents arising in the two-time observables can be found. Assuming the validity of local scale invariance for the relatively large values of  $z$  (usually,  $z \approx 5-7$  [8]) found in many studies of spin glasses and hence the explicit form (4) for the autoresponse function<sup>8</sup>, we shall show that for large times

$$\begin{aligned} \chi''(\omega, t) &= \chi''_1(\omega) + \omega^a \chi''_2(\omega t) + \mathcal{O}(t^{-\lambda_R/z}) \\ \chi'(\omega, t) &= \chi'_1(\omega) + \omega^a \chi'_2(\omega t) + \mathcal{O}(t^{-\lambda_R/z}) \end{aligned} \quad (8)$$

and we shall calculate the scaling functions  $\chi''_2$  and  $\chi'_2$  explicitly in section 2. For the asymptotics of these scaling functions we expect  $\chi'_2(y) \sim y^{-b'}$  and  $\chi''_2(y) \sim y^{-b''}$  for  $y \gg 1$  and obtain the relation

$$b' = b'' = a - a'. \quad (9)$$

This should be compared to the experimentally found scaling (7) of  $\chi''_{\text{age}}(\omega t)$  and similarly for  $\chi'_{\text{age}}(\omega t)$ . In section 3 we compare these results with Monte Carlo data from the 3D Ising spin glass with a binary distribution of the couplings and discuss to what extent the scaling relations (9) and the explicit scaling functions agree with existing experimental data. We conclude in section 4. In appendix A we derive equation (4) from (an extension of) local scale invariance and in appendix B we revisit the OJK approximation of coarsening kinetics and derive (5).

## 2. Scaling of the alternating susceptibility

We now analyse the frequency-dependent scaling of the alternating susceptibility. For notational simplicity, we concentrate first on  $\chi''(\omega, t)$  as given by equation (6). In order to make the scaling behaviour explicit, we must convert this into a more convenient form which can be done as follows [38]. We observe that the time difference  $\tau = t - u$  plays a central role since depending on its value either an equilibrium behaviour or else an ageing behaviour is obtained. Specifically, it can be shown [39] that there is a timescale  $t_p \sim t^\zeta$  with  $0 < \zeta < 1$  on which the transition between the two regimes occurs such that  $R(t, s) \simeq R_{\text{eq}}(t - s)$  for  $t - s \lesssim t_p$  and  $R(t, s) = s^{-1-a} f_R(t/s)$  as given in equation (4) for  $t - s \gtrsim t_p$ .<sup>9</sup> In addition, one measures for  $u \approx t$  the response with respect to a change in the initial conditions and then instead of (4) one expects  $R \approx R_{\text{ini}}(t) \sim t^{-\lambda_R/z}$  [2]. We must therefore introduce a further timescale  $t_\varepsilon$  such that  $t - t_\varepsilon = \mathcal{O}(1)$ . Changing variables and then splitting the integral into three terms corresponding to these three regimes, we have

<sup>8</sup> Our recent study of the critical 3D/4D Ising spin glass [13] showed that the form of the measured thermoremanent magnetization  $\rho(t, s) = \int_0^s du R(t, u)$  agrees with the prediction of local scale invariance for  $t/s \lesssim 20$ , which is also in the sector encountered in  $\omega t$ -scaling.

<sup>9</sup> Explicitly,  $\zeta = 4/(d+2)$  in the  $d$ -dimensional spherical model [39].

$$\begin{aligned}
\chi''(\omega, t) &= \int_0^t d\tau R(t, t - \tau) \sin \omega \tau \\
&= \int_0^{t_p} d\tau R(t, t - \tau) \sin \omega \tau + \int_{t_p}^{t_e} d\tau R(t, t - \tau) \sin \omega \tau \\
&\quad + \int_{t_e}^t d\tau R(t, t - \tau) \sin \omega \tau \\
&\simeq \int_0^{t_p} d\tau R_{\text{eq}}(\tau) \sin \omega \tau + t^{-a} \int_{t_p/t}^{t_e/t} dv f_R \left( \frac{1}{1-v} \right) \frac{\sin \omega t v}{(1-v)^{1+a}} \\
&\quad + t^{-\lambda_R/z} \int_{t_e}^t d\tau c_0 \sin \omega \tau \\
&= \chi_1''(\omega) + t^{-a} \int_0^1 dv f_R \left( \frac{1}{1-v} \right) \frac{\sin \omega t v}{(1-v)^{1+a}} + \mathcal{O}(t^{-\lambda_R/z}). \tag{10}
\end{aligned}$$

In the third line, we used the asymptotic forms of  $R(t, s)$  as described above. This means that the crossover between the equilibrium and the ageing regimes is assumed to be very rapid. In the last line, we restricted ourselves to the long-time limit  $t \rightarrow \infty$ . Here, the function  $\chi_1''(\omega)$  merely depends on the equilibrium form of the linear response  $R_{\text{eq}}(t, s)$ .

In this way (and analogously for  $\chi'$ ) the scaling form (8) is obtained. This derivation also shows that the often-found stationary term in the integrated response [3, 40, 4, 31, 32, 8, 41, 34, 37] does not require the separation of a similar ‘stationary’ part in the response function  $R(t, s)$  itself.

We now analyse the second term in the above expression for  $\chi''$ . Using the explicit form (4) for the scaling function  $f_R$ , we have

$$\chi''(\omega, t) = \chi_1''(\omega) + f_0 t^{-a} S + \mathcal{O}(t^{-\lambda_R/z}) \tag{11}$$

where expansion of the sine followed by termwise integration gives

$$\begin{aligned}
S &:= \int_0^1 dv (1-v)^{-1-a+\lambda_R/z} v^{-1-a'} \sin(\omega t v) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} B \left( 2n+1-a', \frac{\lambda_R}{z} - a \right) (\omega t)^{2n+1} \\
&= B \left( 1-a', \frac{\lambda_R}{z} - a \right) \omega t \\
&\quad \times {}_2F_3 \left( \frac{1-a'}{2}, \frac{2-a'}{2}; \frac{3}{2}, \frac{1-a-a'}{2} + \frac{\lambda_R}{2z}, \frac{2-a-a'}{2} + \frac{\lambda_R}{2z}; -\frac{\omega^2 t^2}{4} \right) \tag{12}
\end{aligned}$$

where  ${}_2F_3$  is a hypergeometric function,

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = \int_0^1 du u^{z-1} (1-u)^{w-1} \tag{13}$$

is Euler’s beta function and the identity  $\Gamma(2z) = 2^{2z-1} \Gamma(z)\Gamma(z+1/2)/\sqrt{\pi}$  was also used.

We proceed to extract the leading behaviour for large values of the scaling variable  $y = \omega t$ . Recall first the asymptotic expansion for  $x \rightarrow +\infty$  [42]:

$${}_2F_3(a, b; c, d, e; -x) \simeq \left[ Ax^{-a} + Bx^{-b} + Cx^{\Delta/2} \cos \left( 2\sqrt{x} + \frac{\pi}{2} \Delta \right) \right] (1 + \mathcal{O}(x^{-1})) \tag{14}$$

with  $\Delta = a + b - c - d - e + \frac{1}{2}$  and where the constants  $A, B, C$  are given by

$$\begin{aligned} A &= \frac{\Gamma(c)\Gamma(d)\Gamma(e)}{\Gamma(b)} \frac{\Gamma(b-a)}{\Gamma(c-a)\Gamma(d-a)\Gamma(e-a)} \\ B &= \frac{\Gamma(c)\Gamma(d)\Gamma(e)}{\Gamma(a)} \frac{\Gamma(a-b)}{\Gamma(c-b)\Gamma(d-b)\Gamma(e-b)} \\ C &= \frac{\Gamma(c)\Gamma(d)\Gamma(e)}{\sqrt{\pi}\Gamma(a)\Gamma(b)}. \end{aligned} \quad (15)$$

Inserting this into the expression for  $S$ , we find

$$S \simeq \bar{A} (\omega t)^{a'-a} + \bar{B} (\omega t)^{a'-a-1} + \bar{C} (\omega t)^{-\lambda_R/z} \sin\left(\omega t + \frac{\pi}{2}(a - \lambda_R/z)\right) \quad (16)$$

where the constants  $\bar{A}, \bar{B}, \bar{C}$  are proportional to  $A, B, C$ . The second term in (16) is always non-leading. The other two terms, however, will determine the functional form of the scaling function for  $y = \omega t$  sufficiently large. The treatment of  $\chi'$  is analogous.

We can summarize the content of this section by listing the scaling functions which occur in (8), together with their leading behaviour as  $y \rightarrow \infty$ :

$$\begin{aligned} \chi_2''(y) &= f_0 B \left(1 - a', \frac{\lambda_R}{z} - a\right) y^{1-a} \\ &\quad \times {}_2F_3\left(\frac{1-a'}{2}, \frac{2-a'}{2}; \frac{3}{2}, \frac{1-a-a'}{2} + \frac{\lambda_R}{2z}, \frac{2-a-a'}{2} + \frac{\lambda_R}{2z}; -\frac{y^2}{4}\right) \\ &\simeq f_0 \frac{\pi}{2} \left[\cos\left(\frac{\pi a'}{2}\right) \Gamma(1+a')\right]^{-1} y^{a'-a} \\ &\quad + f_0 \Gamma\left(\frac{\lambda_R}{z} - a\right) y^{-\lambda_R/z} \sin\left(y + \frac{\pi}{2}[a - \lambda_R/z]\right) \end{aligned} \quad (17)$$

$$\begin{aligned} \chi_2'(y) &= f_0 B \left(-a', \frac{\lambda_R}{z} - a\right) y^{-a} \\ &\quad \times {}_2F_3\left(\frac{-a'}{2}, \frac{1-a'}{2}; \frac{1}{2}, \frac{-a-a'}{2} + \frac{\lambda_R}{2z}, \frac{1-a-a'}{2} + \frac{\lambda_R}{2z}; -\frac{y^2}{4}\right) \\ &\simeq -f_0 \frac{\pi}{2} \left[\sin\left(\frac{\pi a'}{2}\right) \Gamma(1+a')\right]^{-1} y^{a'-a} \\ &\quad + f_0 \Gamma\left(\frac{\lambda_R}{z} - a\right) y^{-\lambda_R/z} \cos\left(y + \frac{\pi}{2}[a - \lambda_R/z]\right). \end{aligned} \quad (18)$$

We see that there appear terms which decrease monotonically with  $y$  but that there are also oscillating terms. They are described by different exponents and must be extracted by a different experimental set-up. The oscillating terms follow the oscillations of the external field and the decrease of the oscillation amplitude gives direct access to the exponent  $\lambda_R/z$ . On the other hand, in many experiments the data are averaged over one or several periods of the external field. For  $y$  sufficiently large, the contribution of the oscillating term in equations (17), (18) vanishes after averaging and then only a simple algebraic component remains, which permits one to extract the exponent  $a - a'$ . For period-averaged data or else if  $\lambda_R/z \geq a - a'$ , the leading behaviour for large arguments is

$$\chi_2'(y) \sim \chi_2''(y) \sim y^{a'-a} \quad (19)$$

and the scaling relations (9) follow.

### 3. Tests

#### 3.1. Numerical simulations

We now compare the theory of the alternating susceptibility developed in the previous section with numerical simulations performed on the critical three-dimensional Ising spin glass. The Hamiltonian of the model is given by

$$\mathcal{H} = - \sum_{(i,j)} J_{i,j} \sigma_i \sigma_j \quad (20)$$

where the nearest-neighbour couplings  $J_{i,j}$  are random variables taken from a binary distribution, i.e.  $J_{i,j} = \pm 1$  with equal probability. The classical Ising spins  $\sigma_i = \pm 1$  characterize the local magnetization at the sites  $i$  of a simple cubic lattice. In the past, numerical studies [43, 44] investigated the static critical properties of this model which undergoes a continuous phase transition at the temperature  $T_c \approx 1.19$  (setting  $k_B = 1$ ). Recently, we studied the ageing behaviour of this critical spin glass [13]. There we found clear evidence of a power-law scaling in the two-time correlation and integrated response functions. Here we shall need the exponent estimates  $a = 0.060(4)$  and  $\lambda_R/z = 0.38(2)$  obtained from a scaling analysis of the thermoremanent magnetization.

In order to study the alternating susceptibility far from equilibrium we prepared the system in an uncorrelated initial state (corresponding to an infinite initial temperature) before quenching it to the critical point at time  $t = 0$ . At the same time an external oscillating magnetic field

$$h(t) = h_0 \cos \omega t \quad (21)$$

was switched on, with its amplitude fixed at  $h_0 = 0.05$  which is well inside the linear response regime. We consider different values of the angular frequency  $\omega = 2\pi/p$  with  $p$  ranging from 50 to 1600. Typically, systems containing  $50^3$  spins were simulated.

Numerically, the in-phase and the out-of-phase susceptibilities are given by the expressions [45]

$$\begin{aligned} \chi''(\omega, t) &= m(t) \sin \omega t \\ \chi'(\omega, t) &= m(t) \cos \omega t \end{aligned}$$

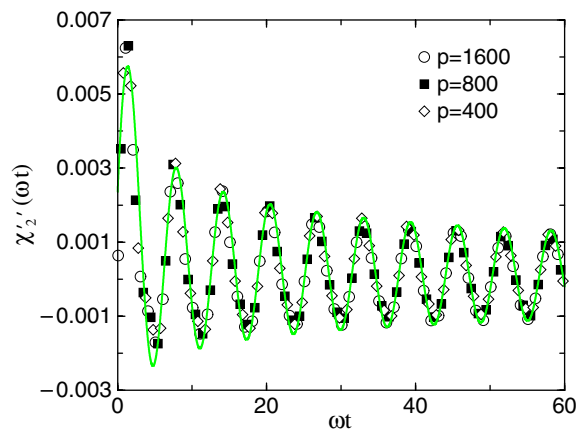
with  $m(t) = \sum_i \sigma_i(t)$ . In order to access the scaling parts of  $\chi''$  and of  $\chi'$ , we must first subtract the equilibrium parts  $\chi''_1$  and  $\chi'_1$ . We therefore carried out longer runs where we let the system equilibrate for typically a few tens of thousands of time steps before switching on the oscillating field. When no changes in the amplitudes of the alternating susceptibilities were observed, we identified these data with the equilibrium parts  $\chi''_1$  and  $\chi'_1$ . The ageing parts of  $\chi''$  and of  $\chi'$  discussed in the following result from averaging over 2500 different runs with different couplings, different initial states and different realizations of the thermal noise.

In figures 1 and 2 we test the expected scaling behaviour of the ageing part; see equation (7):

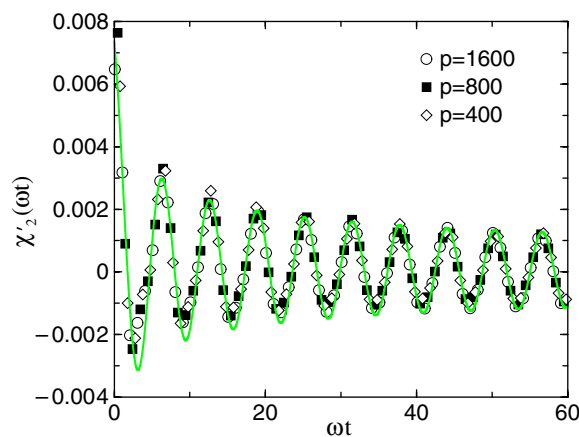
$$\chi''_2 = \chi''_2(\omega t) \quad \text{and} \quad \chi'_2 = \chi'_2(\omega t). \quad (22)$$

For the larger values of  $p$ , corresponding to the smaller values of  $\omega$ , we observe a very good data collapse for both quantities, which furnishes clear evidence in favour of a power-law scaling at  $T = T_c$ . Here we used the value  $a = 0.060(4)$  determined previously from the decay of the thermoremanent magnetization [13]. For smaller values of  $p$ , the collapse is less good, which presumably means that for the corresponding values of  $\omega$  the dynamical scaling regime is not yet reached.





**Figure 1.** Scaling of the dissipative part  $\chi_2''(\omega t)$  of the alternating susceptibility as a function of the scaling variable  $\omega t$  for different angular frequencies  $\omega = 2\pi/p$  with  $p = 1600, 800$  and  $400$ . The full curve is the theoretical prediction (17) with  $f_0 = 0.002$  and  $a' = -0.70$  but it has also been shifted horizontally by  $y \rightarrow y + \Delta y$ , with  $\Delta y = -0.45$ ; see the text. Statistical error bars are smaller than the symbol sizes.



**Figure 2.** The same as figure 1, but now for the dispersive part  $\chi_2'(\omega t)$ . The full curve is the shifted theoretical prediction (18) with  $f_0 = 0.002$  and  $a' = -0.70$ .

The data shown in these two figures can in principle be compared directly with the analytical predictions (17) and (18). We point out, however, that the positions of maxima of  $\chi_2''$  and  $\chi_2'$  are shifted by the amount  $\Delta y \approx -0.45$  when compared with the positions obtained in the analytical treatment of the previous section. The origin of this small shift is not completely clear to us. It is possible, however, that the crossover between the equilibrium and the ageing regimes is not almost instantaneous in contrast to what we assumed in the derivation of equation (10). In any case, we shifted the scaling variable in the analytical curves correspondingly, in order to make a comparison between analytical prediction and numerical data possible.

**Table 1.** Measured values of the exponents  $b''$  and  $b'$  in several glassy materials, obtained using the scaling form (7). Here  $T_g$  stands for the glass transition temperature and  $T$  is the temperature where the data were taken. For  $\text{Fe}_{0.5}\text{Mn}_{0.5}\text{TiO}_3$  and  $\text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4$  the relation  $b' = b''$  was assumed.

Material	$T_g$ (K)	$T$ (K)	$b''$	$b'$	References
$\text{Cu}_{0.5}\text{Co}_{0.5}\text{Cl}_2\text{-FeCl}_3$ -GBIC	3.92(11)	3.25	0.01(4)	0.08(3)	[37]
		3.5	0.017(32)	0.05(2)	
		3.75	0.16(3)	0.20(2)	
		3.85	0.15(3)		
$\text{Fe}_{0.5}\text{Mn}_{0.5}\text{TiO}_3$	20.7	3.95	0.16(4)	0.20(2)	
		15	0.14(3)		[49, 48]
		19	0.14(3)		
$\text{CdCr}_{1.7}\text{In}_{0.3}\text{S}_4$	16.7	12	0.18(3)		[49, 48]
		14	0.18(3)		
$\text{CdCr}_{2x}\text{In}_{2-2x}\text{S}_4$	$x = 0.95$	70	8	0.2	[51, 48]
			67	0.2	
$\text{Pb}(\text{Mg}_{1/3}\text{Nb}_{2/3})\text{O}_3$	$\sim 220$	50	42	0.20	[47, 48]
		$\lesssim 220$	0.17		[40]

As both  $a$  and  $\lambda_R/z$  are known from our earlier investigation [13] the only free parameters in this comparison are the amplitude  $f_0$  and the exponent  $a'$ . Our final estimates are

$$a' = -0.70(3), \quad f_0 = 0.00203(1) \quad (23)$$

and the fit is compared to the data in the figures. These values (23) of the parameters describe consistently both  $\chi_2''$  and  $\chi_2'$ . Although some discrepancies are observed for small values of  $y = \omega t$ , the overall agreement between the simulation and the shifted theoretical prediction is very good. The numerical data therefore support the scaling approach presented in the previous section.

### 3.2. Comparison with experiments

We now turn to a discussion of existing experimental results on the scaling of the alternating susceptibility.

A detailed discussion of a possible scaling of the alternating susceptibility was presented by Suzuki and Suzuki for the short-ranged Ising spin glass  $\text{Cu}_{0.5}\text{Co}_{0.5}\text{Cl}_2\text{-FeCl}_3$  graphite bi-intercalation compound [37]. After a rapid quench to below the glass temperature  $T_g = 3.92(11)$  K, they measure  $\chi''(\omega, t)$  for fixed  $\omega$  and find that their (period-averaged) data are well fitted by the power-law

$$\overline{\chi''}(\omega, t) = \chi_0''(\omega) + A''(\omega)t^{-b''}. \quad (24)$$

While the fitted exponent  $b''$  depends only slightly on  $\omega$ , they further show evidence for a power-law  $A''(\omega) = A_0''\omega^{-\mu''}$  and find that ‘... the value of  $\mu''$  is almost the same as that of  $b''$ ’ [37]. In this way, they arrive at the  $\omega t$ -scaling form

$$\overline{\chi''}(\omega, t) = \chi_0''(\omega) + A_0''(\omega t)^{-b''} \quad (25)$$

and a similar form for  $\overline{\chi'}(\omega, t)$  where ‘...  $b'$  and  $b''$  are of the same order at the same temperature’ [37]. Experimentally measured values of the exponent  $b''$  (and also  $b'$ ) of some materials are collected in table 1. We now compare the experimental results of Suzuki and Suzuki [37] with the theoretical scaling form (8). First, the experimental evidence for a pure  $\omega t$ -scaling indicates that the exponent  $a$  must indeed be very small. Second, when considering

the leading behaviour for  $y = \omega t$  large (their data go up to  $y \lesssim 10^6$  [37]) and recalling that the experimental data are averaged over at least one period of the external field, we can read off

$$b'' = a - a', \quad \mu'' = -a' \quad (26)$$

and the observed [37] near equality  $b'' \approx \mu''$  is again consistent with  $a$  being negligibly small. Third, the available data are consistent with the theoretically required relation  $b' = b''$ . Fourth, the reason for the experimentally observed sudden jump in  $b'$  and  $b''$  for smaller  $T$  is not yet understood but we mention that a similar phenomenon also occurs in certain relaxor ferroelectrics [40, 46].

Similar values of  $b''$  were observed for several other materials, quite independently of the precise physical nature as can be seen from table 1, but the errors are still too large to permit a discussion of the universality of the exponents. However, the experimental data are for many of these systems at least as well described by a logarithmic scaling as expected from the droplet theory [48, 32, 49]. Furthermore, in several systems strong deviations from a simple  $\omega t$ -scaling were also found; see [46]. Finally, we mention that in systems like  $\beta$ -hydroquinol-clathrate [41] or even simple liquids like glycerol [50] a power-law dependence of the form  $\chi''_{\text{age}} \sim t^{-a}$  or  $\chi'_{\text{age}} \sim t^{-a}$  was observed. All in all, it is not yet completely understood what precise conditions are needed such that a clear power-law scaling in  $\omega t$  can be observed.

Lastly, we see that the value  $b'' \simeq 0.7$  obtained from the critical 3D Ising spin glass with binary disorder is very far from the values  $b'' \approx 0.1\text{--}0.2$  found experimentally.

#### 4. Conclusions and discussion

The objective of this work has been to investigate to what extent the ageing behaviour of the dispersive and dissipative parts of the alternating susceptibility in glassy systems may be described in terms of some simple ideas borrowed from the ageing of simple magnets without disorder. These are:

- (i) The clear separation of the stationary and the ageing regimes, which goes into the derivation of equation (10).
- (ii) The hypothesis of a single essential length scale  $L(t) \sim t^{1/z}$  growing algebraically in time.
- (iii) The extension of this dynamical scaling to a local scale invariance which leads to the simple form of equation (4) for the two-time response function  $R(t, s)$ .

Taking these assumptions as a working hypothesis, our results are as follows:

- (i) The often-studied simple  $\omega t$ -scaling (7) in  $\chi''$  and  $\chi'$  should be slightly generalized to the scaling forms (8) or equivalently

$$\chi''(\omega, t) = \chi''_1(\omega) + t^{-a} \bar{\chi}''_2(\omega t) \quad \text{with } \bar{\chi}''_2(y) = y^a \chi''_2(y) \quad (27)$$

and similarly for  $\chi'$ . The smallness of the exponent  $a$  makes it conceivable that the slight ‘subageing’ found in many experiments might be taken into account this way<sup>10</sup>. Tests of this idea on real materials would be welcome.

- (ii) The exponents  $b'$  and  $b''$  defined from  $\chi'_2(y) \sim y^{-b'}$  and  $\chi''_2(y) \sim y^{-b''}$  satisfy the scaling relation

$$b' = b'' = a - a' \quad (28)$$

<sup>10</sup> Working with (27) rather than with (8) has the advantage that for  $y \gg 1$  the first term in the asymptotic expansion only depends on the exponent  $a'$ , while the second one only depends on  $a - \lambda_R/z$ .

which hence relates exponents found in an oscillating field to those describing the two-time response  $R(t, s)$ . Since  $a'$  appears to be a new, independent exponent, this suggests that the exponent  $b' = b''$  should be independent of the exponents  $a, b, \lambda_C/z, \lambda_R/z$  habitually used in describing ageing. This question could be addressed, independently of the hypothesis of local scale invariance and the form (4), through field theory methods as developed in [52].

- (iii) When testing the explicit dispersive and dissipative scaling functions (17) and (18) on the 3D critical Ising spin glass with binary disorder, we found that the form of *both* is reproduced by the data for the following exponents:

$$a = 0.060(4), \quad a' = -0.70(3), \quad \frac{\lambda_R}{z} = 0.38(2) \quad (29)$$

(and with a normalization constant  $f_0 \simeq 0.002$ ) where we took over the values of  $a$  and  $\lambda_R/z$  from our earlier analysis of  $R(t, s)$  in this model [13]. We stress that this agreement is *only* found if the curves (17) and (18) are shifted according to  $y \rightarrow y + \Delta y$  by a constant  $\Delta y \simeq -0.45$ . Even then, for small values of  $y$  the simulated scaling functions show a small but systematic deviation from the theoretical prediction which remains to be understood.

The origin of this shift is unknown to us, but it might be due to a rather slow crossover between the stationary and the ageing regimes which is not captured by equation (10). Understanding this point is an important open problem.

We think that it is remarkable that the ageing dynamics of a system as complicated as a critical spin glass can be captured by our relatively simple hypothesis of a dynamical symmetry.

- (iv) The value  $b'' \simeq 0.7$  that we found for the 3D Ising spin glass (with binary disorder) is far from the values obtained in experiments; see table 1.<sup>11</sup> If the present result were to be confirmed, it might furnish a further hint towards a fundamental difference between simple spin glass models and experimentally realized glassy systems [36].

On the other hand, the theoretically predicted relation  $b' = b''$  is fully consistent with the available experimental results.

## Acknowledgments

We thank G F Mazenko for helpful correspondence and O Petracic for a useful discussion. This work was supported by the Bayerisch-Französisches Hochschulzentrum (BFHZ) and by CINES Montpellier (project pmn2095). MP acknowledges the support by the Deutsche Forschungsgemeinschaft through grant no. PL 323/2.

## Appendix A. The autoresponse function and extended local scale invariance

Local scale transformations are extensions of the dynamical scale transformation  $r \mapsto br$ ,  $t \mapsto b^z t$ , of space–time towards variable rescaling factors  $b = b(t, r)$  such that conformal transformation in time  $t \mapsto (\alpha t + \beta)/(\gamma t + \delta)$  with  $\alpha\delta - \beta\gamma = 1$  is maintained [14]. For any given value of  $z$ , such infinitesimal transformations have been explicitly constructed and shown to furnish a dynamical symmetry of the generalized diffusion equation  $(\partial_t + \partial_r^z) \psi(t, r) = 0$ . Here we are interested in applications to ageing behaviour of the autoresponse function

$$R(t, s) = \left. \frac{\delta \langle \phi(t, r) \rangle}{\delta h(s, r)} \right|_{h=0} = \langle \phi(t, r) \tilde{\phi}(s, r) \rangle \quad (\text{A.1})$$

<sup>11</sup> Our previous investigation of the thermoremanent magnetization [13] did not permit us to determine the value of  $a'$ .

where  $\phi$  is the order parameter,  $h$  the conjugate magnetic field and  $\tilde{\phi}$  the associate response field in the context of the Martin–Siggia–Rose theory; see e.g. [53]. Therefore, we have to restrict consideration to the subalgebra with time translations excluded [14, 15]. Since space translations are included in the set of local scale transformations, the scaling of  $R(t, s)$  will be entirely given in terms of merely two generators which we write as

$$X_0 = -t\partial_t - \frac{x}{z}, \quad X_1 = -t^2\partial_t - \frac{2}{z}(x + \xi)t \quad (\text{A.2})$$

where  $x$  is the scaling dimension of the field on which the generators  $X_n$  act. We have observed earlier [17] for the special case  $z = 2$  that a further constant  $\xi$  can be introduced without changing the commutator relations of the subalgebra of local scale invariance under consideration and now extend this to arbitrary values of  $z$ . Previous treatments of the question had admitted  $\xi = \tilde{\xi} = 0$  from the outset. The covariance of  $R(t, s)$  is now expressed as usual [14, 15]:

$$\begin{aligned} X_0 R &= \left( -t\partial_t - s\partial_s - \frac{x}{z} - \frac{\tilde{x}}{z} \right) R(t, s) = 0 \\ X_1 R &= \left( -t^2\partial_t - s^2\partial_s - \frac{2}{z}(x + \xi)t - \frac{2}{z}(\tilde{x} + \tilde{\xi})s \right) R(t, s) = 0 \end{aligned} \quad (\text{A.3})$$

where  $\tilde{x}$  and  $\tilde{\xi}$  refer to the response field  $\tilde{\phi}$ . To solve these, change the variables into  $u = t - s$  and  $v = t/s$ . Then, with  $R(t, s) = \bar{R}(u, v)$

$$\begin{aligned} \left( u\partial_u + \frac{x + \tilde{x}}{z} \right) \bar{R}(u, v) &= 0 \\ u \left( v\partial_v + \frac{v}{v-1} \frac{x - \tilde{x} + 2\xi}{z} + \frac{1}{v-1} \frac{\tilde{x} - x + 2\tilde{\xi}}{z} \right) \bar{R}(u, v) &= 0. \end{aligned} \quad (\text{A.4})$$

The solution to these is found in factorized form  $\bar{R}(u, v) = f(u)g(v)$  and we find, after having returned to the variables  $t$  and  $s$ ,

$$R(t, s) = r_0 s^{-1-a} \left( \frac{t}{s} - 1 \right)^{-1-a'} \left( \frac{t}{s} \right)^{1+a' - \lambda_R/z} \quad (\text{A.5})$$

where

$$\begin{aligned} 1 + a &= \frac{x + \tilde{x}}{z} \\ 1 + a' &= \frac{x + \tilde{x} + 2\xi + 2\tilde{\xi}}{z} \\ \lambda_R &= 2(x + \xi) \end{aligned} \quad (\text{A.6})$$

which is the form (4) stated in the text. We finally observe that if  $\xi + \tilde{\xi} = 0$ , we do indeed recover  $a = a'$ .

## Appendix B. On the Ohta–Jasnow–Kawasaki approximation

We briefly present the derivation of the scaling of the response function  $R(t, s)$  in a simple analytically tractable scheme which is close in spirit to the Ohta–Jasnow–Kawasaki (OJK) approximation. Following the ideas of Berthier *et al* [21] and of Mazenko [22] which in turn

are based on a calculation by Bray [54], one considers the integrated response in the zero-field-cooled protocol. When perturbing the system by a spatially random field  $h$  of magnitude  $h_0$ , the zero-field-cooled (ZFC) susceptibility reads [21, 54]

$$\chi(t, s) = \frac{\langle h(\mathbf{r})\phi(t, \mathbf{r}) \rangle}{h_0^2} \simeq \sqrt{\frac{2}{\pi}} \frac{\langle h(\mathbf{r})m(t, \mathbf{r}) \rangle}{h_0^2 \sqrt{\langle m^2 \rangle}} \quad (\text{B.1})$$

where  $\phi$  is the order parameter and it is assumed that for late times, one can make the approximation  $\phi \sim \text{sgn}(m)$ . At zero temperature, the auxiliary field  $m$  should satisfy the equation of motion

$$\frac{\partial m}{\partial t} = \nabla^2 m - n_a n_b \nabla_a \nabla_b m + h |\nabla m|. \quad (\text{B.2})$$

In the spirit of the OJK approximation, one makes [54, 2] the simplifications  $n_a n_b \rightarrow \delta_{ab}/d$  (circular average) and  $|\nabla m| \rightarrow \langle (\nabla m)^2 \rangle^{1/2}$ . The equation of motion then becomes [21, 54]  $\partial_t m = D \nabla^2 m + h \langle (\nabla m)^2 \rangle^{1/2}$  with  $D = (d-1)/d$  and, assuming the fields  $m$  and  $h$  to be Gaussian, it is found that [21, equations (19), (20)]

$$\chi(t, s) = \int_s^t du \frac{(Dt)^{d/4}}{(Du)^{(d+2)/4}} \int_{\mathbb{R}^d} d\mathbf{k} e^{-k^2 D(t-u)} \quad (\text{B.3})$$

which is the starting point of our analysis.

Performing the integration in  $\mathbf{k}$ -space, one obtains [22]

$$\chi(t, s) = \text{cste.} \int_s^t du u^{-(d+1)/2} \left(\frac{t}{u}\right)^{d/4} \left(\frac{t}{u} - 1\right)^{-d/2} \stackrel{!}{=} \int_s^t du R(t, u). \quad (\text{B.4})$$

This integral becomes singular near the upper limit  $u \approx t$  and we shall reconsider this below. Before we shall do this, we read off the autoresponse function

$$R(t, u) = u^{-(d+1)/2} f_{\text{OJK}}(t/u), \quad f_{\text{OJK}}(y) = f_0 y^{d/4} (y-1)^{-d/2} \quad (\text{B.5})$$

from which we easily recover, using equation (4), the values of the exponents  $a$ ,  $a'$  and  $\lambda_R$  in the OJK approximation (where  $z = 2$ ) as stated in equation (5). Equation (B.5) is also recovered in the Gaussian theory of phase ordering [23].

A different conclusion was reached by Berthier *et al* [21] although they started from the same point, equation (B.3). They quote  $\chi(t, s) \sim s^{-1/2} F(t/s)$  for  $d > 2$  which would mean  $a = 1/2$ , provided of course that the naive scaling law  $\chi(t, s) = s^{-a} f_\chi(t/s)$  could be used. We must therefore reconsider the singularity in equation (B.4) for  $\chi(t, s)$ . Following [21, 22], one introduces a cut-off parameter  $\Lambda^2$  (which should be sent to zero at the end) and writes instead of (B.3)

$$\chi(t, s) = \int_s^t du \frac{(Dt)^{d/4}}{(Du)^{(d+2)/4}} \int_{\mathbb{R}^d} d\mathbf{k} e^{-k^2 D(t-u+\Lambda^2)}. \quad (\text{B.6})$$

Performing first the integration over  $\mathbf{k}$  and changing variables, this becomes, up to a normalization constant,

$$\chi(t, s) \sim t^{(1-d)/2} \int_{s/t}^1 dv v^{-(d+2)/4} (1-v+\Lambda^2/t)^{-d/2} \quad (\text{B.7})$$

and we must analyse the contribution of the integrand near  $v \approx 1$ . We decompose the domain of integration:  $\int_{s/t}^1 = \int_{s/t}^{1-\varepsilon} + \int_{1-\varepsilon}^1$ . In the first term, we let  $\Lambda^2 \rightarrow 0$ , and  $v \simeq 1$  in the first factor of the second term. Then, for  $d > 2$

$$\begin{aligned} \chi(t, s) &\simeq t^{-(d-1)/2} \int_{s/t}^{1-\varepsilon} dv v^{-(d+2)/4} (1-v)^{-d/2} + t^{-(d-1)/2} \int_{1-\varepsilon}^1 dv (1-v+\Lambda^2/t)^{-d/2} \\ &= t^{-a} F(s/t) + c_1 t^{-1/2} (\Lambda^2)^{1-d/2} + c_2 t^{-a} (\varepsilon + \Lambda^2/t)^{1-d/2} \\ &= s^{-1/2} \cdot \chi_\infty(t/s)^{-1/2} + s^{-a} f_\chi(t/s) \end{aligned} \quad (\text{B.8})$$

where the value  $a = (d - 1)/2$  was used. Here  $c_{1,2}$  and  $\chi_\infty$  are constants and  $F$  and  $f_\chi$  are scaling functions. In this way, not only do we recover the leading term already found in [21], but also we see that the contribution coming from formally integrating the scaling behaviour of  $R(t, s)$  merely gives rise to a subleading correction. A similar argument can be applied to the case  $d = 2$  and produces logarithmic corrections.

Consequently, the mere observation of a scaling law for  $\chi(t, s)$  in the scaling regime  $t, s \rightarrow \infty$  with  $y = t/s$  fixed is not enough for reliably extracting the exponent  $a$ . Indeed, the term expected from naive scaling  $\chi(t, s) \sim s^{-a}$  merely arises as a short-time correction to the leading long-time behaviour of  $\chi(t, s)$ . It has already been pointed out in the context of the Ising model quenched to  $T < T_c$  that a similar dominant term not simply related to the autoresponse exponent  $a$  occurs in the ZFC susceptibility for  $d \leq 3$  [55]. The straightforward use of  $\chi(t, s)$  may hence lead to erroneous values of the exponent  $a$  and it is safer to avoid using  $\chi(t, s)$  altogether.

## References

- [1] Struik L C E 1978 *Physical Ageing in Amorphous Polymers and Other Materials* (Amsterdam: Elsevier)
- [2] Bray A J 1994 *Adv. Phys.* **43** 357
- [3] Cates M E and Evans M R (ed) 2000 *Soft and Fragile Matter* (Bristol: Institute of Physics Publishing)
- [4] Cugliandolo L F 2003 *Slow Relaxation and Non Equilibrium Dynamics in Condensed Matter (Les Houches Session 77)* ed J-L Barrat, J Dalibard, J Kurchan and M V Feigel'man (Berlin: Springer) (*Preprint cond-mat/0210312*)
- [5] Godrèche C and Luck J M 2002 *J. Phys.: Condens. Matter* **14** 1589
- [6] Crisanti A and Ritort F 2003 *J. Phys. A: Math. Gen.* **36** R181
- [7] Henkel M 2004 *Adv. Solid State Phys.* **44** 389
- [8] Kawashima N and Rieger H 2004 *Frustrated Magnetic Systems* ed H Diep (Singapore: World Scientific)
- [9] Calabrese P and Gambassi A 2004 *Preprint cond-mat/0410357*
- [10] Fisher D S and Huse D A 1988 *Phys. Rev. B* **38** 373
- [11] Huse D A 1989 *Phys. Rev. B* **40** 304
- [12] Picone A and Henkel M 2002 *J. Phys. A: Math. Gen.* **35** 5575
- [13] Henkel M and Pleimling M 2005 *Europhys. Lett.* **69** 524  
(Henkel M and Pleimling M 2004 *Preprint cond-mat/0410146*)
- [14] Henkel M 2002 *Nucl. Phys. B* **641** 405
- [15] Henkel M, Pleimling M, Godrèche C and Luck J-M 2001 *Phys. Rev. Lett.* **87** 265701
- [16] Schäfer L 1975 *J. Phys. A: Math. Gen.* **9** 377
- [17] Picone A and Henkel M 2004 *Nucl. Phys. B* **688** 217
- [18] Godrèche C and Luck J M 2000 *J. Phys. A: Math. Gen.* **33** 1151
- [19] Lippiello E and Zanetti M 2000 *Phys. Rev. E* **61** 3369
- [20] Henkel M and Schütz G M 2004 *J. Phys. A: Math. Gen.* **37** 591
- [21] Berthier L, Barrat J L and Kurchan J 1999 *Eur. Phys. J. B* **11** 635
- [22] Mazenko G F 2004 private communication
- [23] Mazenko G F 2004 *Phys. Rev. E* **69** 016114
- [24] Henkel M and Pleimling M 2003 *Phys. Rev. E* **68** 065101(R)
- [25] Abriet S and Karevski D 2004 *Eur. Phys. J. B* **41** 79
- [26] Henkel M, Picone A and Pleimling M 2004 *Europhys. Lett.* **68** 191
- [27] Abriet S and Karevski D 2004 *Eur. Phys. J. B* **37** 47
- [28] Pleimling M 2004 *Phys. Rev. B* **70** 104401
- [29] Enss T, Henkel M, Picone A and Schollwöck U 2004 *J. Phys. A: Math. Gen.* **37** 10479
- [30] Pleimling M and Gambassi A 2005 *Phys. Rev. B* at press (*Preprint cond-mat/0501483*)
- [31] Dupuis V, Bert F, Bouchaud J-P, Hammann J, Ladieu F, Parker D and Vincent E 2004 *Preprint cond-mat/0406672*
- [32] Jönsson P E, Yoshino H, Nordblad P, Agura Katori H and Ito A 2002 *Phys. Rev. Lett.* **88** 257204
- [33] Jiménez S, Martín-Mayor V, Parisi G and Tarancón A 2003 *J. Phys. A: Math. Gen.* **36** 10755
- [34] Rodríguez G F, Kenning G G and Orbach R 2003 *Phys. Rev. Lett.* **91** 037203
- [35] Picco M, Ricci-Tersenghi F and Ritort F 2001 *Eur. Phys. J. B* **21** 211
- [36] Maiorano A, Marinari E and Ricci-Tersenghi F 2004 *Preprint cond-mat/0409577*

- 
- [37] Suzuki I S and Suzuki M 2003 *Phys. Rev. B* **68** 094424
- [38] Henkel M, Paessens M and Pleimling M 2003 *Europhys. Lett.* **62** 644
- [39] Zippold W, Kühn R and Horner H 2000 *Eur. Phys. J. B* **13** 531
- [40] Colla E V, Chao L K, Weissman M B and Viehland D D 2000 *Phys. Rev. Lett.* **85** 3033
- [41] Rheinstädter M, Knorr K and Rieger H 2004 *Phys. Rev. B* **69** 144427
- [42] Wright E M 1935 *J. Lond. Math. Soc.* **10** 286  
Wright E M 1940 *Proc. Lond. Math. Soc.* **46** 389  
Wright E M 1952 *J. Lond. Math. Soc.* **27** 256 (erratum)
- [43] Mari P O and Campbell I A 2002 *Phys. Rev. B* **65** 184409
- [44] Ballesteros H G, Cruz A, Fernández L A, Martín-Mayor V, Pech J, Ruiz-Lorenzo J J, Tarancón A, Téllez P, Ullod C L and Ungil C 2000 *Phys. Rev. B* **62** 14237
- [45] Andersson J-O, Jonsson T and Mattsson J 1996 *Phys. Rev. B* **54** 9912
- [46] Colla E V, Chao L K and Weissman M B 2001 *Phys. Rev. B* **63** 134107
- [47] Vincent E, Dupuis V, Alba M, Hammann J and Bouchaud J-P 2000 *Europhys. Lett.* **50** 674
- [48] Dupuis V 2002 *Thèse de Doctorat* Paris XI Orsay
- [49] Dupuis V, Vincent E, Bouchaud J-P, Hammann J, Ito A and Katori H A 2001 *Phys. Rev. B* **64** 174204
- [50] Leheny R L and Nagel S R 1998 *Phys. Rev. B* **57** 5154
- [51] Dupuis V, Vincent E, Alba M and Hammann J 2002 *Eur. Phys. J. B* **29** 19
- [52] Pimentel I R, Temesvari T and De Dominicis C 2002 *Phys. Rev. B* **65** 224420  
Temesvari T, De Dominicis C and Pimentel I R 2002 *Eur. Phys. J. B* **25** 361
- [53] Janssen H K 1992 *From Phase Transitions to Chaos* ed G Györgyi *et al* (Singapore: World Scientific) p 68
- [54] Bray A J 1997 *ICTP Summer School 'Statistical Physics of Frustrated Systems' (August 1997)*  
[http://www.ictp.trieste.it/~pub\\_off/sci-abs/smr1003/index.html](http://www.ictp.trieste.it/~pub_off/sci-abs/smr1003/index.html)
- [55] Henkel M, Paessens M and Pleimling M 2004 *Phys. Rev. E* **69** 056109